Partial Differential Equations: Midterm Exam

Aletta Jacobshal 01, Friday 4 March 2016, 14:00 - 16:00

Duration: 2 hours

- Solutions should be complete and clearly present your reasoning.
- 10 points are "free". There are 4 questions and the total number of points is 100. The midterm grade is the total number of points divided by 10.
- Do not forget to very clearly write your **full name** and **student number** on the envelope.
- Do not seal the envelope.

Question 1 (20 points)

Consider the first order partial differential equation

$$yu_x + e^x u_y = 0, \tag{1}$$

where $u = u(x, y)$.

(a) (14 points) Find the general solution of Eq. (1).

(b) (6 points) Find the solution of Eq. (1) with the auxiliary condition $u(0, y) = y^4$.

Question 2 (25 points)

Consider the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < \ell$ with the boundary conditions $u_x(0, t) = 0$ and $u(\ell, t) = 0$.

(a) (7 points) Show that if we consider separated solutions of the form $u(x, t) = X(x)T(t)$ then we get the differential equations $-X'' = \lambda X$ and $T' = -ik\lambda T$. What are the boundary conditions satisfied by $X(x)$?

(b) (12 points) It is given that the eigenvalues are $\lambda_n = \beta_n^2 > 0$ where

$$\beta_n = \frac{\pi}{\ell}\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3, \ldots.$$

Solve the differential equations for $X(x)$ and $T(t)$ for the given eigenvalues and write the general solution for $u(x, t)$.

(c) (6 points) Find the solution $u(x, t)$ if it satisfies the initial condition

$$u(x, 0) = \frac{1}{2}\cos\left(\frac{3\pi x}{2\ell}\right) - \frac{3}{8}\cos\left(\frac{7\pi x}{2\ell}\right).$$

Question 3 (20 points)

Consider the second order partial differential equation

$$2u_{xx} - u_{xy} - u_{yy} = 0. \tag{2}$$

where $u = u(x, y)$.

(a) (5 points) Classify the partial differential equation (2) as elliptic, hyperbolic, or parabolic.

(b) (15 points) Find a linear coordinate transformation $(x, y) \to (s, t)$ such that Eq. (2) reduces to the form $u_{st} = 0$; express x and y in terms of s and t. Hint: factorize the second order operator corresponding to the given equation as the product of two first order operators.

Question 4 (25 points)

Consider the eigenvalue problem $-X'' = \lambda X$ for $-\pi < x < \pi$ with boundary conditions $X(-\pi) = -X(\pi)$ and $X'(-\pi) = -X'(\pi)$.

(a) (6 points) Prove that $\lambda = 0$ is not an eigenvalue.

(b) (7 points) Prove that there are no negative eigenvalues.

(c) (12 points) Compute the positive eigenvalues and the corresponding eigenfunctions for this problem.

End of the exam (Total: 90 points)